OPEN STUDENT FOUNDATION

CHAPTER 5 | **Std 12 : MATHS** | Date : 22/02/24

PRACTICE SHEET DAY 4

Section A

- **Write the answer of the following questions. [Each carries 2 Marks]** **[26]**

1. Examine the following functions for continuity : $f(x) = \dfrac{x^2 - 25}{x+5}$, $x \neq -5$

2. Discuss the continuity of the function f, where f is defined by : $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

3. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is a continuous function ?

4. Find $\dfrac{dy}{dx}$ in the following : $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$, $-\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$

5. Find $\dfrac{dy}{dx}$ in the following : $y = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$, $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$

6. Differentiate the following w.r.t. x : $\sqrt{e^{\sqrt{x}}}$, $x > 0$

7. Differentiate the following w.r.t. x : $\cos(\log x + e^x)$, $x > 0$

8. Differentiate the function w.r.t. x : $(\log x)^{\cos x}$

9. Differentiate the function w.r.t. x : $(x\cos x)^x + (x \sin x)^{\frac{1}{x}}$

10. Find $\dfrac{dy}{dx}$ of the function : $(\cos x)^y = (\cos y)^x$

11. Find $\dfrac{dy}{dx}$ of the function : $y^x = x^y$

12. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\dfrac{dy}{dx}$. $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$

13. Differentiate w.r.t. x : $(5x)^{3\cos 2x}$

Section B

- **Write the answer of the following questions. [Each carries 3 Marks]** **[27]**

14. For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$? What about continuity at $x = 1$?

15. Differentiate the functions with respect to x : $\dfrac{\sin(ax+b)}{\cos(cx+d)}$

16. Differentiate the function with respect to x : $2\sqrt{\cot(x^2)}$

17. Differentiate the function w.r.t. x : $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

18. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$

19. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

20. Find the second order derivative of the function : log (log x)

21. Differentiate w.r.t. x : $0 < x < \frac{\pi}{2}$, $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

22. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that $\frac{dy}{dx} = -\frac{-1}{(1+x)^2}$.

Section C

- **Write the answer of the following questions. [Each carries 4 Marks]** **[36]**

23. Differentiate the function w.r.t. x : $\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

24. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$.

25. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

26. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$.

27. If $y = 500\,e^{7x} + 600\,e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49\,y$.

28. If $y = (\tan^{-1} x)^2$ show that $(x^2+1)^2\,y_2 + 2x(x^2+1)\,y_1 = 2$.

29. If $y = e^{a\cos^{-1}x}$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$. Where $-1 \le x \le 1$.

30. If $\cos y = x\cos(a+y)$, with $\cos a \ne \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

31. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Section A

- **Write the answer of the following questions. [Each carries 2 Marks]** **[26]**

1. Examine the following functions for continuity : $f(x) = \dfrac{x^2 - 25}{x + 5}, x \neq -5$

➥ $f(x) = \dfrac{x^2 - 25}{x + 5}, x \neq -5$

Let $g(x) = x^2 - 5$ and $h(x) = x + 5$ be two real functions.

Both are polynomial functions and hence both are continuous functions.

But $f = \dfrac{g}{h}$ is not defined for $x = -5$

Hence, f is continuous $\forall x \in$ R but not for $x = -5$

2. Discuss the continuity of the function f, where f is defined by : $f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$

➥ At $x = 1$,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3 = 3$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4 = 4$

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

$\therefore$ The function $f(x)$ is dicontinuous at $x = 1$.

At $x = 3$,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4 = 4$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5 = 5$

$\therefore \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore$ The function $f(x)$ is dicontinuous at $x = 1$.

3. Determine if f defined by $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ is a continuous function ?

➥ $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right)$

Now, $x \to 0$

$\Rightarrow x \neq 0$

$\Rightarrow -1 \le \sin\frac{1}{x} \le 1$

$\Rightarrow -x^2 \le x^2\sin\frac{1}{x} \le x^2$

$\Rightarrow \lim_{x\to 0} (-x^2) \le \lim_{x\to 0} \left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} (x^2)$

$\Rightarrow 0 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le 0$

$\Rightarrow \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ (By Sandwich theorem)

$\therefore \lim_{x\to 0} f(x) = f(0)$

$\therefore f$ is continuous at $x = 0$.

$\therefore$ The function f is continuous for all $x \in$ R.

4. Find $\frac{dy}{dx}$ in the following : $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

➡ $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\therefore y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$= \tan^{-1} (\tan 3\theta)$ ....(i)

Now $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

$\therefore \tan\left(-\frac{1}{\sqrt{3}}\right) < \tan\theta < \tan\left(\frac{1}{\sqrt{3}}\right)$

$\therefore -\frac{\pi}{6} < \theta < \frac{\pi}{6}$

$\therefore -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$

(i) $\Rightarrow y = \tan^{-1}(\tan 3\theta) = 3\theta$

$\therefore y = 3 \tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2}$

5. Find $\frac{dy}{dx}$ in the following : $y = \sin^{-1}\left(2x\sqrt{1 - x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

➡ $y = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$

Let $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$\therefore\ y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\therefore\ y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\therefore\ y = \sin^{-1}(2\sin\theta \cdot \cos\theta)$

$\therefore\ y = \sin^{-1}(\sin 2\theta)$ ....(i)

Now $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

$\therefore \sin\left(-\frac{\pi}{4}\right) < \sin\theta < \sin\frac{\pi}{4}$

$\therefore -\frac{\pi}{4} < \theta < \frac{\pi}{4}$

$\therefore -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

(i) $\Rightarrow\ y = \sin^{-1}(\sin 2\theta)$

$\therefore\ y = 2\theta = 2\sin^{-1}x$

$\therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

6. Differentiate the following w.r.t. x : $\sqrt{e^{\sqrt{x}}}$, $x > 0$

➥ $y = \sqrt{e^{\sqrt{x}}} = \left(e^{\sqrt{x}}\right)^{\frac{1}{2}}$

On differentiating both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sqrt{x}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(e^{\sqrt{x}}\right)^{\frac{1}{2}-1} \cdot \frac{d}{dx}\left(e^{\sqrt{x}}\right)$$

$$= \frac{1}{2}\left(e^{\sqrt{x}}\right)^{-\frac{1}{2}} \cdot e^{\sqrt{x}}\ \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\left(e^{\sqrt{x}}\right)^{\frac{1}{2}}}\, e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{4}\,\frac{e^{\sqrt{x}}}{\sqrt{xe^{\sqrt{x}}}},\ x > 0$$

7. Differentiate the following w.r.t. x : $\cos(\log x + e^x)$, $x > 0$

➥ $y = \cos(\log x + e^x)$

On differentiating both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos(\log x + e^x)\right]$$

$$= -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x)$$

$= -\sin(\log x + e^x)\left[\frac{d}{dx}(\log x) + \frac{d}{dx}e^x\right]$

$= -\sin(\log x + e^x)\left[\frac{1}{x} + e^x\right]$

8. Differentiate the function w.r.t. x : $(\log x)^{\cos x}$

➥ $y = (\log x)^{\cos x}$

Taking logarithm on both sides,

$\therefore \log y = \log[(\log x)^{\cos x}]$

$\therefore \log y = \cos x \cdot \log(\log x)$

On differentiating both sides w.r. to x,

$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\cos x \cdot \log(\log x)]$

$\therefore \frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}(\log(\log x)) + \log(\log x) \cdot \frac{d}{dx}(\cos x)$ ($\because$ Using product rule)

$\therefore \frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x)(-\sin x)$

$\therefore \frac{dy}{dx} = y\left[\frac{\cos x}{x \cdot \log x} - \sin x \cdot \log(\log x)\right]$

$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \cdot \log x} - \sin x \cdot \log(\log x)\right]$

9. Differentiate the function w.r.t. x : $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

➥ $y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Let $u = (x\cos x)^x$ and $v = (x\sin x)^{\frac{1}{x}}$

$\therefore y = u + v$

Now take $u = (x\cos x)^x$

Taking logarithm on both sides,

$\therefore \log u = x \cdot \log(x\cos x)$

Differentiating both sides w.r. to x,

$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}[\log(x\cos x)] + \log(x\cos x) \cdot \frac{d}{dx}(x)$

$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x\cos x}\frac{d}{dx}(x\cos x) + \log(x\cos x)(1)$

$\therefore \frac{1}{u}\frac{du}{dx} = \frac{1}{\cos x}[x(-\sin x) + \cos x(1)] + \log(x\cos x)$

$\therefore \frac{du}{dx} = u\,[-x\tan x + 1 + \log(x\cos x)]$

$\therefore \frac{du}{dx} = (x\cos x)^x\,[1 - x\tan x + \log(x\cos x)]$ ...(i)

And $v = (x \sin x)^x$

Taking logarithm on both sides,

$$\therefore \log v = \frac{1}{x} \cdot \log(x \sin x)$$

Differentiating both sides w.r. to x,

$$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{1}{x} \cdot \frac{d}{dx}\left(\log(x \sin x)\right) + \log(x \cdot \sin x)\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{1}{x} \cdot \frac{1}{x \sin x} \cdot \frac{d}{dx}(x \sin x) + \log(x \sin x)\left(-\frac{1}{x^2}\right)$$

$$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2 \sin x}(x \cos x + \sin x) - \frac{\log(x \sin x)}{x^2}$$

$$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{x \cot x + 1 - \log(x \sin x)}{x^2}$$

$$\therefore \frac{dv}{dx} = v\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right]$$

$$\therefore \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}}\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right] \text{.(ii)}$$

Now $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = (x \cos x)^x\left[1 - x \tan x + \log(x \cos x)\right] + (x \sin x)^{\frac{1}{x}}\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right]$$

($\because$ From results (i) and (ii))

10. Find $\frac{dy}{dx}$ of the function : $(\cos x)^y = (\cos y)^x$

➡ $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides,

$$\therefore y \cdot \log(\cos x) = x \cdot \log(\cos y)$$

Differentiating both sides w.r. to x,

$$\therefore y \cdot \frac{d}{dx}\left(\log(\cos x)\right) + \log(\cos x)\frac{d}{dx}(y) = x \cdot \frac{d}{dx}\left(\log(\cos y)\right) + \log(\cos y)\frac{d}{dx}(x)$$

$$\therefore y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{\cos y}(-\sin y) \cdot \frac{dy}{dx} + \log(\cos y) \text{ (1)}$$

$$\therefore \frac{dy}{dx}\left[\log(\cos x) + x \tan y\right] = \log(\cos y) + y \tan x$$

$$\therefore \frac{dy}{dx} = \frac{\log(\cos y) + y \tan x}{\log(\cos x) + x \tan y}$$

11. Find $\frac{dy}{dx}$ of the function : $y^x = x^y$

➡ $y^x = x^y$

Taking logarithm on both sides,

$$\therefore x \cdot \log y = y \cdot \log x$$

Differentiating both sides w.r. to x,

$$\therefore x \cdot \frac{d}{dx}(\log y) + \log y \cdot \frac{d}{dx}(x) = y \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(y)$$

$$\therefore x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot (1) = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}\left[\frac{x}{y} - \log x\right] = \frac{y}{x} - \log y$$

$$\therefore \frac{dy}{dx}\left[\frac{x - y \cdot \log x}{y}\right] = \frac{y - x \cdot \log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}\left[\frac{y - x \log y}{x - y \log x}\right]$$

12. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$

➡ $x = \cos\theta - \cos 2\theta$

$$\therefore \frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$$

$y = \sin\theta - \sin 2\theta$

$$\therefore \frac{dy}{d\theta} = \cos\theta - 2\cos 2\theta$$

$$\text{Now } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$$

$$= \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

13. Differentiate w.r.t. x : $(5x)^{3\cos 2x}$

➡ $y = (5x)^{3\cos 2x}$

Taking logarithm on both sides, we have

$$\therefore \log y = 3\cos 2x \cdot \log(5x)$$

$$\therefore \log y = 3\cos 2x\,[\log 5 + \log x]$$

Differentiating w.r. to x on both side, we get

$$\therefore \frac{1}{y}\frac{dy}{dx} = 3\cos 2x \cdot \frac{d}{dx}[\log 5 + \log x] + [\log 5 + \log x] \cdot \frac{d}{dx}(3\cos 2x)$$

$$= 3\cos 2x\left[0 + \frac{1}{x}\right] + [\log 5 + \log x] \cdot [-6\sin 2x]$$

$$= \frac{3\cos 2x}{x} - 6\sin 2x \cdot \log 5x$$

$$\therefore \frac{dy}{dx} = y\left[\frac{3\cos 2x}{x} - 6\sin 2x \cdot \log 5x\right]$$

$$= (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \cdot \log 5x\right]$$

Section B

- **Write the answer of the following questions. [Each carries 3 Marks]** **[27]**

14. For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$? What about continuity at $x = 1$?

➥ $f(x) = \begin{cases} \lambda(x^2 - 2x), & x \le 0 \\ 4x + 1, & x > 0 \end{cases}$ is continious at $x = 0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lambda(x^2 - 2x)$$

$$= \lim_{h \to 0} \lambda[(0 - h)^2 - 2(0 - h)]$$

$$= \lim_{h \to 0} \lambda[h^2 + 2h] = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 4x + 1$$

$$= \lim_{h \to 0} 4(0 + h) + 1$$

$$= \lim_{h \to 0} 4h + 1$$

$$= 1$$

The function $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$\therefore$ 0 = 1 which is not possible.

$\therefore$ For no value of λ, $f(x)$ is continuous at $x = 0$.

At $x = 1$,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x + 1 = 5$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 4x + 1 = 5$$

$\therefore$ $f(x)$ is continuous at $x = 1$ for any value of λ.

15. Differentiate the functions with respect to x : $\dfrac{\sin(ax + b)}{\cos(cx + d)}$

➥ $y = \dfrac{\sin(ax + b)}{\cos(cx + d)}$

Let $u = \sin(ax + b)$ and $v = \cos(cx + d)$

$$\therefore \frac{du}{dx} = \frac{d}{dx} \sin(ax + b)$$

$$= \cos(ax + b) \cdot \frac{d}{dx}(ax + b)$$

$$= \cos(ax + b)\left[\frac{d}{dx}(ax) + \frac{d}{dx}(b)\right]$$

$= \cos(ax + b)\ [a \cdot 1 + 0]$

$= a \times \cos(ax + b)$ ....(i)

$v = \cos(cx + d)$

$\therefore \dfrac{dv}{dx} = \dfrac{d}{dx}\cos(cx + d)$

$= -\sin(cx + d) \cdot \dfrac{d}{dx}(cx + d)$

$= -\sin(cx + d)\left[\dfrac{d}{dx}(cx) + \dfrac{d}{dx}(d)\right]$

$= -\sin(cx + d)\ [c(1) + 0]$

$= -c\ \sin(cx + d)$ ....(ii)

Now $y = \dfrac{u}{v}$

Using division rule we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right)$$

$$= \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{du}}{v^2}$$

Using result (i), (ii) and substitute the value of u and v, we get,

$$\frac{dy}{dx} = \frac{\cos(cx+d) \cdot a\cos(ax+b) - \sin(ax+b)[-c\sin(cx+d)}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)\cos(cx+d) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(ax+b)\cos(cx+d)}{\cos^2(cx+d)} + \frac{c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$= a \cdot \cos(ax + b)\ \sec(cx + d) + c \cdot \sin(ax + b) \cdot \tan(cx + d) \cdot \sec(cx + d)$

16. Differentiate the function with respect to x : $2\sqrt{\cot(x^2)}$

➥ $y = 2\sqrt{\cot(x^2)}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[2\left(\cot(x^2)\right)^{\frac{1}{2}}\right]$$

$$= 2 \cdot \frac{1}{2}\left(\cot(x^2)\right)^{\frac{1}{2}-1} \cdot \frac{d}{dx}\left(\cot(x^2)\right)$$

$$= \left(\cot(x^2)\right)^{-\frac{1}{2}} \cdot \left(-\mathrm{cosec}^2(x^2)\right) \cdot \frac{d}{dx}(x^2)$$

$$= \frac{1}{\sqrt{\cot(x^2)}} \cdot \left(-\mathrm{cosec}^2(x^2)\right) \cdot 2x$$

$$= \frac{-2x\sqrt{\sin(x^2)}}{\sqrt{\cot(x^2)} \cdot \sin^2(x^2)}$$

$$= \frac{-2x}{\sin(x^2)\sqrt{\sin(x^2)\cos(x^2)}}$$

$$= \frac{-2x}{\sin(x^2)\sqrt{\frac{2\sin x^2 \cdot \cos x^2}{2}}}$$

$$= \frac{-2\sqrt{2}x}{\sin(x^2) \cdot \sqrt{\sin 2x^2}}$$

17. Differentiate the function w.r.t. x : $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

➥ $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm on both sides, we have

$$\therefore \log y = \log\left[\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\right]$$

$$\therefore \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\therefore \log y = \frac{1}{2}\{\log((x-1)(x-2)) - \log(x-3)(x-4)(x-5)\}$$

$$\therefore \log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

On differentiating both sides w.r. to x, we get

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}y\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

18. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$

➥ $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$

$$\therefore \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \cdot \frac{1}{2\cos^2\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\cos\frac{t}{2}\sin\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(-\frac{\sin^2 t + 1}{\sin t}\right)$$

$$= \frac{a\cos^2 t}{\sin t} \qquad \text{..........(i)}$$

$y = a \sin t$

$$\therefore \frac{dy}{dt} = a\cos t \qquad \text{..........(ii)}$$

Now $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\cos^2 t}\sin t$ ($\because$ From (i) and (ii))

$= \tan t$

19. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta \cos\theta)$

➥ $x = a(\cos\theta + \theta\sin\theta)$

$$\therefore \frac{dx}{d\theta} = a \cdot \frac{d}{d\theta}(\cos\theta + \theta\sin\theta)$$

$$\therefore \frac{dx}{d\theta} = a[-\sin\theta + \theta\cos\theta + \sin\theta]$$

$$= a\,\theta\cos\theta \qquad \text{.........(i)}$$

$y = a(\sin\theta - \theta\cos\theta)$

$$\therefore \frac{dy}{d\theta} = a \cdot \frac{d}{d\theta}(\sin\theta - \theta\cos\theta)$$

$$= a[\cos\theta - \cos\theta + \theta\sin\theta]$$

$$= a\,\theta\sin\theta \qquad \text{.........(ii)}$$

Now $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\,\theta\sin\theta}{a\,\theta\cos\theta}$ ($\because$ From results (i) and (ii))

$= \tan\theta$

20. Find the second order derivative of the function : $\log(\log x)$

➥ Take $y = \log(\log x)$

Now differentiating w.r. to x on both sides,

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x))$$

$$\therefore \frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x\log x}$$

Again differentiating w.r. to x on both sides,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x\log x}\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{x\log x\,\frac{d}{dx}(1) - 1\,\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$= \frac{0 - \left[x\cdot\frac{1}{x} + \log x\right]}{(x\log x)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-(1+\log x)}{(x\log x)^2}$$

21. Differentiate w.r.t. x : $0 < x < \frac{\pi}{2}$, $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

➥ Take $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$= \cot^{-1}\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}$$

$$= \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}-\sin\frac{x}{2}}\right]$$

$$= \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] \qquad \left(\because 0 < x < \frac{\pi}{2} \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{4}\right)$$

$$\therefore y = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

22. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that $\frac{dy}{dx} = -\frac{-1}{(1+x)^2}$.

➥ $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\therefore x\sqrt{1+y} = -y\sqrt{1+x}$

$\therefore x^2(1+y) = y^2(1+x)$ ($\because$ squaring both sides)

$\therefore (x^2 - y^2) + xy(x-y) = 0$

$\therefore (x+y) + xy = 0$ ($\because$ Devide both sides by $(x-y)$)

$\therefore y(1+x) = -x$

$$\therefore y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x)(-1) - (-x)(0+1)}{(1+x)^2}$$

$$= \frac{-1 - x + x}{(1+x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

Section C

- **Write the answer of the following questions. [Each carries 4 Marks]** **[36]**

23. Differentiate the function w.r.t. x : $\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

➥ $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Let $u = \left(x + \frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

Now $u = \left(x + \frac{1}{x}\right)^x$

Taking logarithm on both sides,

$$\therefore \log u = x \cdot \log\left(x + \frac{1}{x}\right)$$

Differentiating both sides w.r. to x,

$$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}\left(\log\left(x + \frac{1}{x}\right)\right) + \log\left(x + \frac{1}{x}\right) \cdot \frac{d}{dx}(x)$$

$$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x + \frac{1}{x}}\frac{d}{dx}\left(x + \frac{1}{x}\right) + \log\left(x + \frac{1}{x}\right) \cdot (1)$$

$$\therefore \frac{1}{u}\frac{du}{dx} = \frac{x^2}{x^2+1}\left(1 - \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right)$$

$$\therefore \frac{du}{dx} = u\left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right]$$

$$\therefore \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right] \quad ....(i)$$

$v = x^{\left(1+\frac{1}{x}\right)}$

Taking logarithm on both sides,

$$\therefore \log v = \left(1 + \frac{1}{x}\right) \cdot \log x$$

Differentiating both sides w.r. to x,

$$\therefore \frac{1}{v}\frac{dv}{dx} = \left(1 + \frac{1}{x}\right) \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left(1 + \frac{1}{x}\right)$$

$$\therefore \frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right)\cdot\frac{1}{x} + \log x\left(0 - \frac{1}{x^2}\right)$$

$$\therefore \frac{dv}{dx} = v\left[\frac{x+1}{x^2} - \frac{\log x}{x^2}\right]$$

$$\therefore \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right] \qquad \text{......(ii)}$$

Now $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

From results (i) and (ii) we have,

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]$$

24. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$.

➥ Here $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$

$$\therefore x = \left(a^{\sin^{-1}t}\right)^{\frac{1}{2}},\ y = \left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$$

Now $x^2 = a^{\sin^{-1}t}$, $y^2 = a^{\cos^{-1}t}$

$$\therefore x^2 \cdot y^2 = a^{\sin^{-1}t + \cos^{-1}t}$$

$$\therefore x^2 \cdot y^2 = (a)^{\frac{\pi}{2}}$$

Now differentiate w.r. to x on both sides,

$$2x \cdot y^2 + x^2(2y)\frac{dy}{dx} = 0$$

$$\therefore x^2 y \frac{dy}{dx} = -xy^2$$

$$\therefore \frac{dy}{dx} = -\frac{xy^2}{yx^2}$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}$$

25. If x and y are connected parametrically by the equations, without eliminating the parameter, Find $\frac{dy}{dx}$. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

➥ $$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$$

$$\therefore \frac{dx}{dt} = \frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}(\sin^3 t) - \sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\left(\sqrt{\cos 2t}\right)^2}$$

$$= \frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cos t - \sin^3 t\cdot\left(\frac{1}{2\sqrt{\cos 2t}}\cdot(-2\sin 2t)\right)}{}$$

$\cos 2t$

$$= \frac{3\sin^2 t \cdot \cos t \cdot \sqrt{\cos 2t} + \frac{\sin^3 t \cdot \sin 2t}{\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{3\sin^2 t \cdot \cos t \cdot \cos 2t + \sin^3 t \cdot \sin 2t}{(\cos 2t)^{\frac{3}{2}}} \quad ..(i)$$

$$y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dt} = \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}(\cos^3 t) - \cos^3 t \frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

$$= \frac{-\sqrt{\cos 2t} \cdot 3\cos^2 t \sin t + \cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} 2\sin 2t}{(\sqrt{\cos 2t})^2}$$

$$= \frac{-3\cos^2 t \cdot \sin t \cos 2t + \cos^3 t \cdot \sin 2t}{(\cos 2t)^{\frac{3}{2}}} \quad .(ii)$$

Now $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$= \frac{\cos^3 t \cdot \sin 2t - 3\cos^2 t \sin t \cos 2t}{3\sin^2 t \cdot \cos t \cdot \cos 2t + \sin^3 t \cdot \sin 2t} \quad \text{(From (i) and (ii))}$$

$$= \frac{\sin 2t - 3\tan t \cdot \cos 2t}{3\tan^2 t \cdot \cos 2t + \tan^3 t \cdot \sin 2t} \quad \text{(Dividing N}^r \text{ and D}^r \text{ by } \cos^2 t)$$

$$= \frac{2\tan t - 3\tan t\,(1 - \tan^2 t)}{3\tan^2 t\,(1 - \tan^2 t) + \tan^3 t \cdot 2\tan t} \quad \left(\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}, \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

using these formulas

$$= \frac{-(1 - 3\tan^2 t)}{3\tan t - \tan^3 t} = -\cot 3t \quad \left(\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

26. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$.

➥ $y = 3\cos(\log x) + 4\sin(\log x)$

$$y_1 = \frac{dy}{dx} = -3\sin(\log x) \cdot \left(\frac{1}{x}\right) + 4\cos(\log x) \cdot \left(\frac{1}{x}\right)$$

$$\therefore y_1 = \frac{-3\sin(\log x) + 4\cos(\log x)}{x}$$

$$\therefore x y_1 = -3\sin(\log x) + 4\cos(\log x)$$

Again differentiating w.r. to x, we get

$$\therefore x y_2 + y_1 = -3\cos(\log x) \cdot \frac{1}{x} - 4\sin(\log x) \cdot \frac{1}{x}$$

$$\therefore x y_2 + y_1 = \frac{-3\cos(\log x) - 4\sin(\log x)}{x}$$

$$\therefore x^2 y_2 + x y_1 = -(3\cos(\log x) + 4\sin(\log x)$$

$$\therefore x^2 y_2 + x y_1 = -y$$

$$\therefore x^2 y_2 + x y_1 + \text{y} = 0$$

27. If $y = 500\ e^{7x} + 600\ e^{-7x}$, show that $\dfrac{d^2y}{dx^2} = 49\ y$.

➥ $y = 500\ e^{7x} + 600\ e^{-7x}$

Now differentiating w.r. to x on both sides,

$$\frac{dy}{dx} = 500 \times 7e^{7x} - 600 \times 7 \times e^{-7x}$$

Again differentiating w.r. to x, we have

$$\therefore \frac{d^2y}{dx^2} = 500 \times 49\, e^{7x} + 600 \times 49\, e^{-7x}$$

$$= 49\ [500\ e^{7x} + 600\ e^{-7x}]$$

$$\therefore \frac{d^2y}{dx^2} = 49\ y$$

28. If $y = (\tan^{-1} x)^2$ show that $(x^2 + 1)^2\ y_2 + 2x(x^2 + 1)\ y_1 = 2$.

➥ $y = (\tan^{-1} x)^2$

Now differentiating w.r. to x on both sides,

$$\therefore \frac{dy}{dx} = y_1 = 2\tan^{-1} x \cdot \frac{1}{1 + x^2}$$

$$\therefore (1 + x^2)\ y_1 = 2\ \tan^{-1} x$$

Squaring on both sides,

$$\therefore (1 + x^2)^2\ {y_1}^2 = 4\ (\tan^{-1}x)^2$$

$$\therefore (1 + x^2)^2 \cdot {y_1}^2 = 4y$$

Again differentiating w.r. to x on both sides,

$$\therefore (1 + x^2)^2 \cdot 2y_1y_2 + 2(1 + x^2)\ 2x \cdot {y_1}^2 = 4y_1$$

$$\therefore (1 + x^2)^2\ y_2 + 2x\ (1 + x^2)\ y_1 = 2$$

($\because$ Dividing by $2y_1 \neq 0$)

29. If $y = e^{a\cos^{-1}x}$ show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0$. Where $-1 \le x \le 1$.

➥ $y = e^{a\cos^{-1}x}$

Differentiating w.r. to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{a\cos^{-1}x}\right)$$

$$\therefore \frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \frac{d}{dx}\left(e^{a\cos^{-1}x}\right)$$

$$\therefore \frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \left(\frac{-1 \cdot a}{\sqrt{1 - x^2}}\right)$$

$$\therefore \sqrt{1 - x^2} \cdot \frac{dy}{dx} = -ay. \quad \left(\because e^{a\cos^{-1}x} = y\right)$$

Squaring both sides,

$$(1 - x^2)\left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

Again differentiating w.r. to x, we get

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = 2a^2 y\frac{dy}{dx}$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2 y \left(\because \text{Divide both sides by } 2\frac{dy}{dx}\right)$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0.$$

30. If $\cos y = x\cos(a + y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

➥ $\cos y = x\cos(a + y)$

Now differentiate w.r. to x on both sides,

$$\therefore -\sin y\frac{dy}{dx} = -x\sin(a+y)\frac{dy}{dx} + \cos(a+y)$$

$$\therefore \left(x\sin(a+y) - \sin y\right)\frac{dy}{dx} = \cos(a+y)$$

$$\therefore \frac{dy}{dx} = \frac{\cos(a+y)}{x\sin(a+y) - \sin y}$$

$$= \frac{\cos(a+y)}{\left(\frac{\cos y}{\cos(a+y)}\right)\sin(a+y) - \sin y} \quad \left(\begin{matrix}\because \cos y = x\cos(a+y)\ (\text{given}) \\ \because x = \frac{\cos y}{\cos(a+y)}\end{matrix}\right)$$

$$= \frac{\cos^2(a+y)}{\sin(a+y)\cos y - \cos(a+y)\sin y}$$

$$= \frac{\cos^2(a+y)}{\sin(a+y-y)} \quad (\because \text{Using } \sin(\alpha - \beta))$$

$$\therefore \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$ Which is required result.

31. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that $\dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

➥ $(x - a)^2 + (y - b)^2 = c^2$

Differentiate w.r. to x, we have

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\therefore (x-a) + (y-b)\frac{dy}{dx} = 0 \qquad \text{........(i)}$$

Again differentiate w.r. to x, we get

$$\therefore 1 + (y - b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\therefore (y - b)\frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{y - b} \qquad \text{........(ii)}$$

$$\text{(i)} \Rightarrow \frac{dy}{dx} = -\left(\frac{x - a}{y - b}\right)$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x - a)^2}{(y - b)^2}$$

$$= \frac{(y - b)^2 + (x - a)^2}{(y - b)^2}$$

$$= \frac{c^2}{(y - b)^2} \qquad \text{........(iii)}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \left[\frac{c^2}{(y - b)^2}\right]^{\frac{3}{2}}$$

$$= \frac{c^3}{(y - b)^3} \qquad \text{........(iv)}$$

$$\text{(ii)} \Rightarrow \frac{d^2y}{dx^2} = -\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{(y - b)}$$

$$= -\frac{c^2 / (y - b)^2}{y - b} \qquad (\because \text{From (iii)})$$

$$= -\frac{c^2}{(y - b)^3} \qquad \text{........(v)}$$

From results (iv) and (v),

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\frac{c^3}{(y - b)^3}}{\frac{-c^2}{(y - b)^3}}$$

$$= -c$$

Which is constant independent of a and b.